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A generalization of a modular identity of Rogers

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ABSTRACT

In a handwritten manuscript published with his lost notebook, Ramanujan stated without proofs forty identities for the Rogers–Ramanujan functions. Most of the elementary proofs given for these identities are based on Schröter-type theta function identities in particular, the identities of L.J. Rogers. We give a generalization of Rogers's identity that also generalizes similar formulas of H. Schröter, and of R. Blecksmith, J. Brillhart, and I. Gerst. Applications to modular equations, Ramanujan's identities for the Rogers–Ramanujan functions as well as new identities for these functions are given.

1. Introduction

The Rogers–Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{1.1}$$

where $(a;q)_0 := 1$, and for $n \geqslant 1$,

$$(a;q)_n := \prod_{k=0}^{n-1} \left(1 - aq^k\right).$$

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These functions satisfy the famous Rogers–Ramanujan identities [6,8], [7, pp. 214–215]

$$G(q)=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}\quad\text{and}\quad H(q)=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \tag{1.2}$$

where

$$(a;q)_\infty:=\lim_{n\to\infty}(a;q)_n,\quad |q|<1.$$

In a handwritten manuscript published with his lost notebook, Ramanujan stated without proofs forty identities for the Rogers–Ramanujan functions. The simplest yet the most elegant is the following identity which was proved by Rogers [9]

$$H(q)G\big(q^{11}\big)-q^2G(q)H\big(q^{11}\big)=1. \tag{1.3}$$

D. Bressoud [5], in his PhD thesis, generalized Rogers's method, developed similar identities and proved fifteen identities from Ramanujan's list of forty. Here and throughout the manuscript by Rogers's lemma we mean its generalization given by Bressoud. The generalization we give here directly implies or greatly simplifies the proofs given by Bressoud and others that are based on Schröter-type theta function identities. A detailed history of Ramanujan's forty identities can be found in [2].

The rest of the paper is organized as follows. The preliminary results are given in Section 2. In the following section, we give the generalization of Rogers's lemma, Theorem 3.1 and its corollaries. As applications we provide new modular equations as theta function identities and new identities for the Rogers–Ramanujan functions. We also obtain as a special case a formula of Blecksmith, Brillhart, and Gerst [4] that provides a representation for a product of two fairly general theta functions as a certain sum of products of pairs of theta functions. This formula, in turn, generalizes formulas of Schröter [1, pp. 65–72], which have been enormously useful in establishing many of Ramanujan's modular equations [1]. In Section 4, we consider a special case of our formula, Theorems 4.3 and 4.4, where we employ the quintuple product identity, and as special cases we provide proofs for the following three identities of Ramanujan whose only known proofs are by Biagioli [3], who used the theory of modular forms. Let $\chi(q):=(-q;q^2)_\infty$.

Entry 1.1.

$$\frac{G(q^{19})H(q^4)-q^3G(q^4)H(q^{19})}{G(q^{76})H(-q)+q^{15}G(-q)H(q^{76})}=\frac{\chi(-q^2)}{\chi(-q^{38})}. \tag{1.4}$$

Entry 1.2.

$$\frac{G(q^2)G(q^{33})+q^7H(q^2)H(q^{33})}{G(q^{66})H(q)-q^{13}H(q^{66})G(q)}=\frac{\chi(-q^3)}{\chi(-q^{11})}. \tag{1.5}$$

Entry 1.3.

$$\frac{G(q^3)G(q^{22})+q^5H(q^3)H(q^{22})}{G(q^{11})H(q^6)-qG(q^6)H(q^{11})}=\frac{\chi(-q^{33})}{\chi(-q)}. \tag{1.6}$$

2. Definitions and preliminary results

We first recall Ramanujan's definition for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

For convenience, we also define

$$f_k(a,b) = \begin{cases} f(a,b) & \text{if } k \equiv 0 \pmod{2}, \\ f(-a,-b) & \text{if } k \equiv 1 \pmod{2}. \end{cases} \tag{2.2}$$

Basic properties satisfied by $f(a,b)$ include [1, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{2.3}$$

$$f(1,a) = 2f\left(a,a^3\right), \tag{2.4}$$

$$f(-1,a) = 0, \tag{2.5}$$

and if u is an integer,

$$f(a,b) = a^{u(u+1)/2} b^{u(u-1)/2} f\left(a(ab)^u, b(ab)^{-u}\right). \tag{2.6}$$

The identity (2.6) will be used many times in the sequel. For convenience, we record the following special case corresponding to $u = 1$

$$f_k\left(q^{-x}, q^y\right) = (-1)^k q^{-x} f_k\left(q^x, q^{y-2x}\right). \tag{2.7}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [1, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.8}$$

The three most important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty, \tag{2.9}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.10}$$

and

$$f(-q) := f\left(-q,-q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty =: q^{-1/24}\eta(\tau), \tag{2.11}$$

where $q = \exp(2\pi i\tau)$, $\operatorname{Im}\tau > 0$, and η denotes the Dedekind eta-function. The product representations in (2.9)–(2.11) are special cases of (2.8). Also, after Ramanujan, define

$$\chi(q) := \left(-q;q^2\right)_\infty. \tag{2.12}$$

Using (2.8) and (2.11), we can rewrite the Rogers–Ramanujan identities (1.2) in the forms

$$G(q)=\frac{f(-q^2,-q^3)}{f(-q)} \quad \text{and} \quad H(q)=\frac{f(-q,-q^4)}{f(-q)}. \tag{2.13}$$

We shall use the famous quintuple product identity, which, in Ramanujan's notation, takes the form [1, p. 80, Entry 28(iv)]

$$\frac{f(-a^2,-a^{-2}q)}{f(-a,-a^{-1}q)}=\frac{1}{f(-q)}\left\{f\left(-a^3q,-a^{-3}q^2\right)+af\left(-a^{-3}q,-a^3q^2\right)\right\}, \tag{2.14}$$

where a is any complex number.

The function $f(a,b)$ also satisfies a useful addition formula. For each nonnegative integer n, let

$$U_n:=a^{n(n+1)/2}b^{n(n-1)/2} \quad \text{and} \quad V_n:=a^{n(n-1)/2}b^{n(n+1)/2}.$$

Then [1, p. 48, Entry 31]

$$f(U_1,V_1)=\sum_{r=0}^{n-1}U_r f\left(\frac{U_{n+r}}{U_r},\frac{V_{n-r}}{U_r}\right). \tag{2.15}$$

Two special cases of (2.15) which we frequently use are

$$\varphi(q)=\varphi\left(q^4\right)+2q\psi\left(q^8\right) \tag{2.16}$$

and

$$\psi(q)=f\left(q^6,q^{10}\right)+qf\left(q^2,q^{14}\right). \tag{2.17}$$

3. Generalization of Roger's lemma

Let m be an integer and α, β, p and λ be positive integers such that

$$\alpha m^2+\beta=p\lambda. \tag{3.1}$$

Let δ, ε be integers. Further let l and t be real and x and y be nonzero complex numbers. Recall that the general theta functions f, f_k are defined by (2.1) and (2.2). With the parameters defined this way, we set

$$\begin{aligned} &R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)\\ &:=\sum_{\substack{k=0\\ n=2k+t}}^{p-1}(-1)^{\varepsilon k}y^k q^{\{\lambda n^2+p\alpha l^2+2\alpha nml\}/4}f_\delta\left(xq^{(1+l)p\alpha+\alpha nm},x^{-1}q^{(1-l)p\alpha-\alpha nm}\right)\\ &\quad\times f_{\varepsilon p+m\delta}\left(x^{-m}y^pq^{p\beta+\beta n},x^my^{-p}q^{p\beta-\beta n}\right). \end{aligned} \tag{3.2}$$

We are now ready to state the main theorem of this section.

Theorem 3.1.

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)=\sum_{u,\,v=-\infty}^{\infty}(-1)^{\delta v+\varepsilon u}x^v y^u q^{T/4}, \tag{3.3}$$

where

$$T:=\lambda U^2+2\alpha mUV+p\alpha V^2 \tag{3.4}$$

$$=\lambda\left(U+\frac{\alpha m}{\lambda}V\right)^2+\frac{\alpha\beta}{\lambda}V^2 \tag{3.5}$$

$$=p\alpha\left(V+\frac{m}{p}U\right)^2+\frac{\beta}{p}U^2, \tag{3.6}$$

with $U:=2u+t$ *and* $V:=2v+l$.

Proof. From (3.2) and (2.2), we have

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)=\sum_{\substack{k=0\\ n=2k+t}}^{p-1}\sum_{r,\,s=-\infty}^{\infty}(-1)^{\varepsilon k+\delta r+(\varepsilon p+m\delta)s}y^k x^r\left(x^{-m}y^p\right)^s q^{T_1}, \tag{3.7}$$

where

$$T_1:=\left\{\lambda n^2+p\alpha l^2+2\alpha nml\right\}/4+p\alpha r^2+(lp\alpha+\alpha nm)r+p\beta s^2+\beta ns. \tag{3.8}$$

Fix s and let $r=ms+v$. We find that

$$\varepsilon k+\delta r+(\varepsilon p+m\delta)s\equiv\varepsilon(ps+k)+\delta v \pmod 2 \tag{3.9}$$

and

$$y^k x^r\left(x^{-m}y^p\right)^s=x^v y^{ps+k}. \tag{3.10}$$

In (3.8), we set $r=ms+v$ and use (3.1), and after some tedious algebra, we conclude that

$$4T_1=\lambda(2ps+n)^2+p\alpha(2v+l)^2+2\alpha m(2v+l)(2ps+n). \tag{3.11}$$

Recall that $n=2k+t$. Letting $u:=ps+k$, $U:=2u+t$, and $V:=2v+l$, we find that

$$2ps+n=U \quad\text{and}\quad 4T_1=\lambda U^2+2\alpha mUV+p\alpha V^2=T. \tag{3.12}$$

Eqs. (3.5) and (3.6) are easily verified by (3.1). Next, we return to (3.7) and use (3.9)–(3.12) to conclude that

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)$$
$$=\sum_{\substack{k=0\\ u=ps+k}}^{p-1}\sum_{v,s=-\infty}^{\infty}(-1)^{\delta v+\varepsilon u}x^v y^u q^{T/4}=\sum_{u,v=-\infty}^{\infty}(-1)^{\delta v+\varepsilon u}x^v y^u q^{T/4}. \qquad\square \tag{3.13}$$

From (3.5) and (3.6) we deduce the following corollary:

Corollary 3.2.

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)=R(\delta,\varepsilon,t,l,1,\alpha\beta,\alpha m,\lambda,p\alpha,y,x). \tag{3.14}$$

Corollary 3.3. *Let α_1,β_1,m_1,p_1 be another set of parameters such that $\alpha_1 m_1^2+\beta_1=p_1\lambda$, $\alpha\beta=\alpha_1\beta_1$ and $\lambda\mid(\alpha m-\alpha_1 m_1)$. Set*

$$a:=\frac{\alpha m-\alpha_1 m_1}{\lambda}. \tag{3.15}$$

Then,

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)=R\big(\varepsilon,\delta+a\varepsilon,l,t+al,\alpha_1,\beta_1,m_1,p_1,\lambda,xy^{-a},y\big). \tag{3.16}$$

Proof. Replace u by $u-av$ in (3.13). $\square$

Theorem 3.1 and Corollary 3.3 give a generalization of Rogers's lemma which is the special case when m is odd, $x=y=1$, $l=0$, $t=1$ and $\delta\equiv\varepsilon p+m\delta \pmod 2$. Observe that

$$\alpha p\lambda-\alpha_1 p_1\lambda=\alpha m^2+\alpha\beta-\alpha_1 m_1^2-\alpha_1\beta_1=(\alpha m-\alpha_1 m_1)(\alpha m+\alpha_1 m_1). \tag{3.17}$$

Therefore if λ is prime then the condition $\lambda\mid(\alpha m-\alpha_1 m_1)$ is always satisfied (replace m_1 by $-m_1$ if necessary). Corollary 3.2 is new and we now consider some applications of it to Ramanujan's identities for the Rogers–Ramanujan functions and to theta function identities.

Ramanujan's identities for the Rogers–Ramanujan functions are given in terms of the function

$$U(r,s):=\begin{cases} G(q^r)G(q^s)+q^{(s+r)/5}H(q^r)H(q^s) & \text{if } s+r\equiv 0 \pmod 5,\\ H(q^r)G(q^s)-q^{(s-r)/5}G(q^r)H(q^s) & \text{if } s-r\equiv 0 \pmod 5.\end{cases} \tag{3.18}$$

As an example [5],

$$U(1,19)=\frac{1}{4\sqrt{q}}\chi^2\big(q^{1/2}\big)\chi^2\big(q^{19/2}\big)-\frac{1}{4\sqrt{q}}\chi^2\big(-q^{1/2}\big)\chi^2\big(-q^{19/2}\big)-\frac{q^2}{\chi^2(-q)\chi^2(-q^{19})}. \tag{3.19}$$

Here we prove (3.19) and provide similar identities. It will be convenient to work with the function

$$u(r,s):=\begin{cases} g(q^r)g(q^s)+q^{(s+r)/5}h(q^r)h(q^s) & \text{if } s+r\equiv 0 \pmod 5,\\ h(q^r)g(q^s)-q^{(s-r)/5}g(q^r)h(q^s) & \text{if } s-r\equiv 0 \pmod 5,\end{cases} \tag{3.20}$$

where

$$g(q):=f\big(-q^2,-q^3\big)\quad\text{and}\quad h(q):=f\big(-q,-q^4\big). \tag{3.21}$$

By (2.13), we have that $u(r,s)=f(-q^r)f(-q^s)U(r,s)$ and by (2.9) and (2.10), Eq. (3.19) can be written as

$$4qu(2,38)=\varphi(q)\varphi\big(q^{19}\big)-\varphi(-q)\varphi\big(-q^{19}\big)-4q^5\psi\big(q^2\big)\psi\big(q^{38}\big). \tag{3.22}$$

From Corollary 3.2, and by (2.16), we find that

$$\begin{aligned}2qu(2,38) &= R(0,1,0,1,1,19,1,5,4,1,1)\\ &= R(1,0,1,0,1,19,1,4,5,1,1)\\ &= qf\big(1,q^8\big)f\big(q^{76},q^{76}\big) - 2q^5f\big(q^2,q^6\big)f\big(q^{38},q^{114}\big) + q^{19}f\big(q^4,q^4\big)f\big(1,q^{152}\big)\\ &= \frac{1}{2}\big(\varphi(q)\varphi\big(q^{19}\big) - \varphi(-q)\varphi\big(-q^{19}\big) - 4q^5\psi\big(q^2\big)\psi\big(q^{38}\big)\big),\end{aligned} \tag{3.23}$$

which is (3.22).

From Corollary 3.2 with the following choice of parameters

$$\begin{aligned}R(0,1,0,1,3,17,1,5,4,1,1) &= R(1,0,1,0,1,51,3,4,15,1,1),\\ R(0,1,0,1,1,51,3,5,12,1,1) &= R(1,0,1,0,3,17,1,4,5,1,1),\\ R(0,1,0,1,7,13,1,5,4,1,1) &= R(1,0,1,0,1,91,7,4,35,1,1),\\ R(0,1,0,1,1,91,7,5,28,1,1) &= R(1,0,1,0,7,13,1,4,5,1,1),\\ R(0,1,0,1,9,11,1,5,4,1,1) &= R(1,0,1,0,1,99,9,4,45,1,1),\\ R(0,1,0,1,1,99,9,5,36,1,1) &= R(1,0,1,0,9,11,1,4,5,1,1),\end{aligned}$$

we similarly obtain the following new identities

$$\begin{aligned}4qu(6,34) &= \varphi(q)\varphi\big(q^{51}\big) - \varphi(-q)\varphi\big(-q^{51}\big) - 4q^{13}\psi\big(q^2\big)\psi\big(q^{102}\big),\\ 4q^3u(2,102) &= \varphi\big(q^3\big)\varphi\big(q^{17}\big) - \varphi\big(-q^3\big)\varphi\big(-q^{17}\big) - 4q^5\psi\big(q^6\big)\psi\big(q^{34}\big),\\ 4qu(14,26) &= \varphi(q)\varphi\big(q^{91}\big) - \varphi(-q)\varphi\big(-q^{91}\big) - 4q^{23}\psi\big(q^2\big)\psi\big(q^{182}\big),\\ 4q^5u(2,182) &= 4q^5\psi\big(q^{14}\big)\psi\big(q^{26}\big) - \varphi\big(q^7\big)\varphi\big(q^{13}\big) + \varphi\big(-q^7\big)\varphi\big(-q^{13}\big),\\ 4qu(18,22) &= \varphi(q)\varphi\big(q^{99}\big) - \varphi(-q)\varphi\big(-q^{99}\big) - 4q^{25}\psi\big(q^2\big)\psi\big(q^{198}\big),\\ 4q^5u(2,198) &= 4q^5\psi\big(q^{18}\big)\psi\big(q^{22}\big) - \varphi\big(q^9\big)\varphi\big(q^{11}\big) + \varphi\big(-q^9\big)\varphi\big(-q^{11}\big).\end{aligned}$$

Similar identities exists for $p = 4, 5$, $\lambda \in \{1, 2, 4, 8\}$. We give one example for $\lambda = 8$.

From Corollary 3.2

$$\begin{aligned}2u(6,74) &= 2R(0,1,0,1,3,37,1,5,8,1,1)\\ &= 2R(1,0,1,0,1,111,3,8,15,1,1)\\ &= 2q^2f\big(1,q^{16}\big)f\big(q^{888},q^{888}\big) - 4q^{14}f\big(q^6,q^{10}\big)f\big(q^{666},q^{1110}\big) + 4q^{56}f\big(q^4,q^{12}\big)f\big(q^{444},q^{1332}\big)\\ &\quad - 4q^{126}f\big(q^2,q^{14}\big)f\big(q^{222},q^{1554}\big) + 2q^{222}f\big(q^8,q^8\big)f\big(1,q^{1776}\big)\\ &= \varphi\big(q^2\big)\varphi\big(q^{222}\big) - \varphi\big(-q^2\big)\varphi\big(-q^{222}\big) + 4q^{56}\psi\big(q^4\big)\psi\big(q^{444}\big)\\ &\quad - 4q^{14}\big(f\big(q^6,q^{10}\big)f\big(q^{666},q^{1110}\big) + q^{112}f\big(q^2,q^{14}\big)f\big(q^{222},q^{1554}\big)\big)\\ &= \varphi\big(q^2\big)\varphi\big(q^{222}\big) - \varphi\big(-q^2\big)\varphi\big(-q^{222}\big) + 4q^{56}\psi\big(q^4\big)\psi\big(q^{444}\big)\\ &\quad - 2q^{14}\big(\psi(q)\psi\big(q^{111}\big) + \psi(-q)\psi\big(-q^{111}\big)\big),\end{aligned}$$

where in the last step, we used (2.17).

For $\lambda \in \{3,6\}$, together with φ and ψ the theta functions $f(q,q^5)=\chi(-q)\psi(-q^3)$ and $f(q,q^2)=\frac{\varphi(-q^3)}{\chi(-q)}$ also appear, and so $u(r,s)$ may still be written as sums of eta-quotients.

If $\alpha m^2+\beta=4,6,8,9,12,16,18,24,32,36,48,64,72$ under some parity restriction we obtain two representations as sums of eta-quotients and therefore the resulting identity can be regarded as a modular equation. Most of these modular equations were given by Ramanujan and were later proved using Schröter's formulas [1]. We give one example that seems to be new. By Corollary 3.2, and by (2.16) and (2.17), we have

$$\begin{aligned}
&R(0,0,0,0,5,59,1,8,8,1,1)\\
&\quad= f\left(q^{40},q^{40}\right)f\left(q^{472},q^{472}\right)+2q^8 f\left(q^{30},q^{50}\right)f\left(q^{354},q^{590}\right)+2q^{32} f\left(q^{20},q^{60}\right)f\left(q^{236},q^{708}\right)\\
&\qquad+2q^{72} f\left(q^{10},q^{70}\right)f\left(q^{118},q^{826}\right)+q^{128} f\left(1,q^{80}\right)f\left(1,q^{944}\right)\\
&\quad=\left(\varphi\left(q^{10}\right)\varphi\left(q^{118}\right)+\varphi\left(-q^{10}\right)\varphi\left(-q^{118}\right)\right)/2+2q^{32}\psi\left(q^{20}\right)\psi\left(q^{236}\right)\\
&\qquad+q^8\left(\psi\left(q^5\right)\psi\left(q^{59}\right)+\psi\left(-q^5\right)\psi\left(-q^{59}\right)\right)\\
&\quad=R(0,0,0,0,1,295,5,8,40,1,1)\\
&\quad= f\left(q^{8},q^{8}\right)f\left(q^{2360},q^{2360}\right)+2q^{38} f\left(q^{2},q^{14}\right)f\left(q^{1770},q^{2950}\right)+2q^{148} f\left(q^{4},q^{12}\right)f\left(q^{1180},q^{3540}\right)\\
&\qquad+2q^{332} f\left(q^{6},q^{10}\right)f\left(q^{590},q^{4130}\right)+q^{592} f\left(1,q^{16}\right)f\left(1,q^{4720}\right)\\
&\quad=\left(\varphi\left(q^{2}\right)\varphi\left(q^{590}\right)+\varphi\left(-q^{2}\right)\varphi\left(-q^{590}\right)\right)/2+2q^{148}\psi\left(q^{4}\right)\psi\left(q^{1180}\right)\\
&\qquad+q^{37}\left(\psi(q)\psi\left(q^{295}\right)-\psi(-q)\psi\left(-q^{295}\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\varphi\left(q^{10}\right)\varphi\left(q^{118}\right)+\varphi\left(-q^{10}\right)\varphi\left(-q^{118}\right)+4q^{32}\psi\left(q^{20}\right)\psi\left(q^{236}\right)\\
&\qquad+2q^8\left(\psi\left(q^5\right)\psi\left(q^{59}\right)+\psi\left(-q^5\right)\psi\left(-q^{59}\right)\right)\\
&\quad=\varphi\left(q^{2}\right)\varphi\left(q^{590}\right)+\varphi\left(-q^{2}\right)\varphi\left(-q^{590}\right)+4q^{148}\psi\left(q^{4}\right)\psi\left(q^{1180}\right)\\
&\qquad+2q^{37}\left(\psi(q)\psi\left(q^{295}\right)-\psi(-q)\psi\left(-q^{295}\right)\right).
\end{aligned}$$

Three other identities similar to the one just stated can be obtained by changing the parities of ε and δ. This of course can be duplicated for any other pair whose sum is 64. We now prove the aforementioned formula of Blecksmith, Brillhart, and Gerst [4]. The reformulation we give here can be found in [1, p. 73].

Define, for $\varepsilon \in \{0,1\}$ and $|ab|<1$,

$$f_\varepsilon(a,b)=\sum_{n=-\infty}^{\infty}(-1)^{\varepsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

Theorem 3.4. *Let a, b, c, and d denote positive numbers with $|ab|, |cd|<1$. Suppose that there exist positive integers u, v, and n such that*

$$(ab)^v=(cd)^{u(n-uv)}. \tag{3.24}$$

Let $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, and define $\delta_1, \delta_2 \in \{0,1\}$ by

$$\delta_1 \equiv \varepsilon_1 - u\varepsilon_2 \pmod 2 \quad \text{and} \quad \delta_2 \equiv v\varepsilon_1 + s\varepsilon_2 \pmod 2, \tag{3.25}$$

respectively, where $s = n - uv$. Then, if E denotes any complete residue system modulo n,

$$f_{\varepsilon_1}(a,b) f_{\varepsilon_2}(c,d) = \sum_{r\in E} (-1)^{\varepsilon_2 r} c^{r(r+1)/2} d^{r(r-1)/2} f_{\delta_1}\left(\frac{a(cd)^{u(u+1-2r)/2}}{c^u}, \frac{b(cd)^{u(u+1+2r)/2}}{d^u}\right)$$
$$\times f_{\delta_2}\left(\frac{(b/a)^{v/2}(cd)^{s(n+1-2r)/2}}{c^s}, \frac{(a/b)^{v/2}(cd)^{s(n+1+2r)/2}}{d^s}\right). \tag{3.26}$$

Proof. We replace, without lost of generality, a, b, c and d by xq^a, $x^{-1}q^a$, yq^b, $y^{-1}q^b$, and assume that $\gcd(v, n) = 1$ and that $E = \{0, 1, \dots, n-1\}$. Then, by (3.24), the right-hand side of (3.26) takes the form

$$\sum_{r=0}^{n-1} (-1)^{\varepsilon_2 r} y^r q^{br^2} f_{\delta_1}\left(xy^{-u} q^{\frac{bu}{v}(n-2vr)}, x^{-1}y^u q^{\frac{bu}{v}(n+2vr)}\right) \tag{3.27}$$
$$\times f_{\delta_2}\left(x^{-v}y^{-s} q^{\frac{av}{u}(n-2ur)}, x^v y^s q^{\frac{av}{u}(n+2ur)}\right)$$
$$= R\left(\varepsilon_2, \varepsilon_1 - u\varepsilon_2, 0, 0, \frac{bu}{v}, \frac{av}{u}, v, n, b, x^{-1}y^u, y\right)$$
$$= R\left(\varepsilon_2, \varepsilon_1, 0, 0, a, b, 0, 1, b, x^{-1}, y\right)$$
$$= f_{\varepsilon_1}\left(xq^a, x^{-1}q^a\right) f_{\varepsilon_2}\left(yq^b, y^{-1}q^b\right), \tag{3.28}$$

where we used Corollary 3.3 with the set of variables $\alpha_1 = a$, $\beta_1 = b$, $m_1 = 0$, $p_1 = 1$, $\lambda = b$, and $\alpha_2 = b\frac{u}{v}$, $\beta_2 = a\frac{v}{u}$, $m_1 = v$, $p_1 = n$, $\lambda = b$. $\square$

4. Further extensions of Theorem 3.1

Our next theorems, Theorems 4.3 and 4.4, significantly differ from the previous two theorems and will be used in Section 5 to prove Entries 1.1—1.3. We start with several preliminaries.

Lemma 4.1. *Let l, t and z be integers with $z \in \{-1, 1\}$. Define $\delta_1 := \varepsilon p + m\delta$ and assume that*

$$\varepsilon(p+t) + \delta(l+m) \equiv 1 \pmod 2. \tag{4.1}$$

Then,

$$R1(z, \varepsilon, \delta, l, t, \alpha, \beta, m, p) := R\left(\varepsilon, \delta, l - \frac{zm}{3}, t + \frac{zp}{3}, \alpha, \beta, m, p, \lambda, 1, 1\right)$$
$$= (-1)^{\frac{(z+1)(1+\delta_1)}{2}} q^{\frac{1}{4}\{p\alpha l^2 + p\beta/9\}} f\left(-q^{2p\beta/3}\right)\left\{S1 + (-1)^{\varepsilon t/2} S2\right\}, \tag{4.2}$$

where

$$S1 = \sum_{\substack{n=1 \\ n\equiv t \pmod 2}}^{p-1} (-1)^{\varepsilon(n-t)/2} q^{\frac{1}{4}\{\lambda n^2 + 2\alpha mnl - 2n\beta/3\}} \frac{f(-q^{2\beta n/3}, -q^{2p\beta/3 - 2\beta n/3})}{f_{\delta_1}(q^{\beta n/3}, q^{2p\beta/3 - \beta n/3})}$$
$$\times f_\delta\left(q^{(1+l)p\alpha + \alpha mn}, q^{(1-l)p\alpha - \alpha mn}\right), \tag{4.3}$$

$$S2 = \begin{cases} f_\delta(q^{(1+l)p\alpha}, q^{(1-l)p\alpha}) & \textit{if } t \equiv \delta_1 + 1 \equiv 0 \pmod 2, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.4}$$

Proof. Using the definition (3.2), we find after some algebra that

$$\begin{aligned}&R1(z,\varepsilon,\delta,l,t,\alpha,\beta,m,p)\\&\quad=q^{p\beta/36+p\alpha l^2/4}\sum_{\substack{k=0\\n=2k+t}}^{p-1}(-1)^{\varepsilon k}q^{\frac14\{\lambda n^2+2\alpha mnl+2nz\beta/3\}}f_\delta\big(q^{(1+l)p\alpha+\alpha mn},q^{(1-l)p\alpha-\alpha mn}\big)\\&\qquad\times f_{\delta_1}\big(q^{(1+z/3)p\beta+\beta n},q^{(1-z/3)p\beta-\beta n}\big).\end{aligned}\tag{4.5}$$

Observe that if $t\equiv\tilde t \pmod 2$, then by Theorem 3.1, we have

$$R1(\varepsilon,\delta,l,t,\alpha,\beta,m,p)=(-1)^{\frac{(t-\tilde t)\varepsilon}{2}}R1(\varepsilon,\delta,l,\tilde t,\alpha,\beta,m,p).\tag{4.6}$$

Since (4.6) holds with $R1$ replaced by $S1$ or $(-1)^{\varepsilon t/2}S2$ and $(-1)^{\frac{(z+1)(1+\delta_1)}{2}}$ remains unchanged, we will assume without loss of generality that $t\in\{0,1\}$. We will show that the contribution of the terms with indices k, $1-t\leqslant k\leqslant\lfloor\frac{p-1-t}{2}\rfloor$, and $p-k-t$ can be combined to a single product via the quintuple product identity. The exceptions are clearly those for which

$$k=p-k-t\quad\text{or}\quad p-k-t>p-1,\quad 0\leqslant k\leqslant p-1.\tag{4.7}$$

We will show that these exceptions will make up the sum $(-1)^{\frac{(z+1)(1+\delta_1)}{2}+\varepsilon t/2}q^{p\beta/36+p\alpha l^2/4}\times f(-q^{2p\beta/3})S2$ which can be determined by examining several cases. We will only look at the case where p is even and t is even since the other cases are similar. If p is even and t is even, then by (4.1), δ and $l+m$ are both odd. The exceptions for k are 0 and $p/2$. Observe that if l is an integer and $l_1\equiv l \pmod 2$ then, by (2.6) with $a=(-1)^\delta q^{1-l}$, $b=(-1)^\delta q^{1+l}$ and $u=(l-l_1)/2$, we find that

$$f_\delta\big(q^{1-l},q^{1+l}\big)=(-1)^{\delta(l-l_1)/2}q^{(l_1^2-l^2)/4}f_\delta\big(q^{1-l_1},q^{1+l_1}\big).\tag{4.8}$$

Therefore, by (2.5),

$$f_\delta\big(q^{1-l},q^{1+l}\big)=0\quad\text{if } l\delta\equiv1\pmod 2.\tag{4.9}$$

The contribution of the term with $k=0$ is

$$q^{p\beta/36+p\alpha l^2/4}f_\delta\big(q^{(1+l)p\alpha},q^{(1-l)p\alpha}\big)f_{1+l}\big(q^{2p\beta/3},q^{4p\beta/3}\big).\tag{4.10}$$

When $k=p/2$, the corresponding term has the factor $f_\delta(q^{p\alpha(1+l+m)},q^{p\alpha(1-l-m)})$ which, by (4.9), is zero since $m+l$ and δ are both odd. Therefore, only (4.10) contributes to $S2$ and that this agrees with (4.4) since if l is odd, then, by (4.9) and the fact that δ is odd, the first theta function in (4.10) is identically zero. Next, we look at the contribution of the terms with indices $p-k-t$. Observe that if k is replaced by $p-k-t$ then n is replaced by $2p-n$. By (2.6) with $a=(-1)^\delta q^{(1-l)p\alpha-\alpha m(2p-n)}$, $b=(-1)^\delta q^{(1+l)p\alpha+\alpha m(2p-n)}$ and $u=l+m$, we find that

$$\begin{aligned}&f_\delta\big(q^{(1-l)p\alpha-\alpha m(2p-n)},q^{(1+l)p\alpha+\alpha m(2p-n)}\big)\\&\quad=(-1)^{\delta(l+m)}q^{\alpha m(m+l)(n-p)}f_\delta\big(q^{(1+l)p\alpha+\alpha mn},q^{(1-l)p\alpha-\alpha mn}\big).\end{aligned}\tag{4.11}$$

Similarly, by (2.7), we find that

$$f_{\delta_1}\left(q^{(1+z/3)p\beta+\beta(2p-n)}, q^{(1-z/3)p\beta-\beta(2p-n)}\right)$$
$$= (-1)^{\delta_1} q^{\beta(n-p-zp/3)} f_{\delta_1}\left(q^{(1-z/3)p\beta+\beta n}, q^{(1+z/3)p\beta-\beta n}\right). \tag{4.12}$$

By (4.5), (4.11)–(4.12) and the parity condition (4.1), we find after some algebra that the sum of the terms with indices k, $1-t \leqslant k \leqslant \lfloor \frac{p-1-t}{2} \rfloor$, and $p-k-t$, are

$$S := \sum_{\substack{n=1 \\ n \equiv t \pmod 2}}^{p-1} (-1)^{\varepsilon(n-t)/2} q^{\frac{1}{4}\{\lambda n^2+2\alpha mnl+2zn\beta/3\}} f_\delta\left(q^{(1+l)p\alpha+\alpha mn}, q^{(1-l)p\alpha-\alpha mn}\right)$$
$$\times \left\{ f_{\delta_1}\left(q^{(1+z/3)p\beta+\beta n}, q^{(1-z/3)p\beta-\beta n}\right) + q^{-\beta nz/3}(-1)^{\delta_1+1} f_{\delta_1}\left(q^{(1-z/3)p\beta+\beta n}, q^{(1+z/3)p\beta-\beta n}\right)\right\}. \tag{4.13}$$

Now, we employ the quintuple product identity, (2.14), with q replaced by $q^{2p\beta/3}$ and a replaced by $(-1)^{\delta_1+1}q^{-\beta zn/3}$, and use the fact that z is either 1 or -1, to find that

$$f_{\delta_1}\left(q^{(1+z/3)p\beta+\beta n}, q^{(1-z/3)p\beta-\beta n}\right) + q^{-\beta nz/3}(-1)^{\delta_1+1} f_{\delta_1}\left(q^{(1-z/3)p\beta+\beta n}, q^{(1+z/3)p\beta-\beta n}\right)$$
$$= f\left(-q^{2p\beta/3}\right) \frac{f(-q^{-2\beta nz/3}, -q^{2p\beta/3+2\beta nz/3})}{f_{\delta_1}(q^{-\beta nz/3}, q^{2p\beta/3+\beta nz/3})}. \tag{4.14}$$

Observe that by (2.7),

$$f\left(-q^{2p\beta/3}\right) \frac{f(-q^{-2\beta n/3}, -q^{2p\beta/3+2\beta n/3})}{f_{\delta_1}(q^{-\beta n/3}, q^{2p\beta/3+\beta n/3})}$$
$$= (-1)^{\delta_1+1} q^{-\beta n/3} f\left(-q^{2p\beta/3}\right) \frac{f(-q^{2\beta n/3}, -q^{2p\beta/3-2\beta n/3})}{f_{\delta_1}(q^{\beta n/3}, q^{2p\beta/3-\beta n/3})}. \tag{4.15}$$

By (4.13)–(4.15), we conclude that

$$S = (-1)^{\frac{z+1}{2}(1+\delta_1)} f\left(-q^{2p\beta/3}\right) S1.$$

Moreover, if $\delta_1 \equiv 0 \pmod 2$, then, by (4.4), $S2 = 0$. Therefore, the proof of Lemma 4.1 is complete. □

Lemma 4.2. *Let l and t be integers. Define $\delta_1 := \varepsilon p + m\delta$ and assume that*

$$\varepsilon t + \delta(l+1) \equiv 1 \pmod 2. \tag{4.16}$$

Define

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) := R\left(\varepsilon, \delta, l - \frac{1}{3}, t, \alpha, \beta, m, p, \lambda, 1, 1\right).$$

If $\gcd(m, p) = 1$, *then,*

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = q^{\frac{p\alpha}{36}} f\left(-q^{2p\alpha/3}\right)\{S3 + S4\}, \tag{4.17}$$

where

$$S3 = \sum_{\substack{n=1 \\ n \equiv t \pmod 2}}^{p-1} (-1)^{\varepsilon(n-t)/2} q^{\frac{1}{4}\{\lambda n^2 + 2\alpha mn(l-1/3) + p\alpha l(l-2/3)\}} \frac{f(-q^{2\alpha(nm+lp)/3}, -q^{2p\alpha/3 - 2\alpha(nm+lp)/3})}{f_\delta(q^{\alpha(nm+lp)/3}, q^{2p\alpha/3 - \alpha(nm+lp)/3})}$$
$$\times f_{\delta_1}\big(q^{p\beta+\beta n}, q^{p\beta-\beta n}\big), \tag{4.18}$$

$$S4 = \begin{cases} (-1)^{(l+t\varepsilon)/2} \varphi_{\delta_1}(q^{p\beta}) & \textit{if } t \equiv 0 \pmod 2, \\ 2(-1)^{\frac{m+l+\varepsilon(p-t)}{2}} q^{p\beta/4} \psi(q^{2p\beta}) & \textit{if } p \equiv t \equiv \delta \equiv 1 + m + l \equiv 1 \pmod 2, \\ 0 & \textit{otherwise}. \end{cases} \tag{4.19}$$

The proof of Lemma 4.2 is very similar to that of Lemma 4.1 and so we forego the proof. Observe that if $\gcd(m, p) = d$, then by Corollary 3.3, we have that

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = R2(\varepsilon, \delta, l, t, d\alpha, \beta/d, m/d, p/d). \tag{4.20}$$

Therefore, the assumption $\gcd(m, p) = 1$ does not restrict the applicability of Lemma 4.2.

Theorem 4.3. *Let α, β, m, p, and λ be as before with $\alpha m^2 + \beta = p\lambda$, and let $\varepsilon, \delta, l, t$ be integers with $(1+l)\delta + t\varepsilon \equiv 1 \pmod 2$. Assume further that $3 \mid \alpha m$ and $\gcd(3, \lambda) = 1$. Recall that $R1$ and $R2$ are defined by (4.2) and (4.17). Let α_1, β_1, m_1, and p_1 be another set parameters as in Corollary 3.3 and set $a := (\alpha m - \alpha_1 m_1)/\lambda$. Then,*

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = R1(z, \delta, \varepsilon, l_1, t_1, 1, \alpha\beta, \alpha m, \lambda), \tag{4.21}$$

where $l_1 := t + \alpha mz/3$, $t_1 := l - 1/3 - z\lambda/3$ and $z = \pm 1$ with $z \equiv -\lambda \pmod 3$. Moreover, if $3 \mid \alpha_1 m_1$, then

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = R2(\varepsilon, \delta + a\varepsilon, l, t_2, \alpha_1, \beta_1, m_1, p_1), \tag{4.22}$$

where $t_2 = t + a(l - 1/3)$.

If $3 \mid \beta_1$ and $\gcd(3, \alpha_1 m_1) = 1$, then

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = R1(y, \varepsilon, \delta + a\varepsilon, l_3, t_3, \alpha_1, \beta_1, m_1, p_1), \tag{4.23}$$

where $y = \pm 1$ with $y \equiv m_1 \pmod 3$, $l_3 = l - 1/3 + ym_1/3$, and $t_3 = t + a(l - 1/3) - yp_1/3$.

Proof. The proofs of (4.21)–(4.23) are essentially the same, so we prove (4.21) in detail and give a sketch of the proofs of the latter two. Since $z \equiv -\lambda \pmod 3$ and $3 \mid \alpha m$, $t_1 := l - 1/3 - z\lambda/3$ and $l_1 := t + \alpha mz/3$ are both integers. Moreover,

$$\delta(\lambda + t_1) + \varepsilon(l_1 + \alpha m) \equiv \delta(\lambda + l + 1 + \lambda) + \varepsilon(t + \alpha m + \alpha m) \equiv \delta(l+1) + \varepsilon t \equiv 1 \pmod 2. \tag{4.24}$$

By (4.17), (3.14), and by (4.24), we have

$$\begin{aligned} R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) &= R(\varepsilon, \delta, l - 1/3, t, \alpha, \beta, m, p, \lambda, 1, 1) \\ &= R(\delta, \varepsilon, t, l - 1/3, 1, \alpha\beta, \alpha m, \lambda, p\alpha, 1, 1) \\ &= R(\delta, \varepsilon, t + \alpha mz/3 - \alpha mz/3, l - 1/3 - \lambda z/3 + \lambda z/3, 1, \alpha\beta, \alpha m, \lambda, p\alpha, 1, 1) \\ &= R(\delta, \varepsilon, l_1 - \alpha mz/3, t_1 + \lambda z/3, 1, \alpha\beta, \alpha m, \lambda, p\alpha, 1, 1) \\ &= R1(z, \delta, \varepsilon, l_1, t_1, 1, \alpha\beta, \alpha m, \lambda), \end{aligned}$$

which is (4.21). Now assume $3 \mid \alpha_1 m_1$. By (4.17) and (3.16), we find that

$$\begin{aligned}R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) &= R(\varepsilon,\delta,l-1/3,t,\alpha,\beta,m,p,\lambda,1,1)\\ &= R(\varepsilon,\delta+a\varepsilon,l-1/3,t+a(l-1/3),\alpha_1,\beta_1,m_1,p_1,\lambda,1,1)\\ &= R2(\varepsilon,\delta+a\varepsilon,l,t_2,\alpha_1,\beta_1,m_1,p_1),\end{aligned}$$

which proves (4.22). Now assume $\gcd(3,\alpha_1 m_1)=1$ and $3 \mid \beta_1$. We have

$$\begin{aligned}R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) &= R(\varepsilon,\delta,l-1/3,t,\alpha,\beta,m,p,\lambda,1,1)\\ &= R(\varepsilon,\delta+a\varepsilon,l-1/3,t+a(l-1/3),\alpha_1,\beta_1,m_1,p_1,\lambda,1,1)\\ &= R(\varepsilon,\delta+a\varepsilon,l-1/3+ym_1/3-ym_1/3,t+a(l-1/3)-yp_1/3\\ &\quad +yp_1/3,\alpha_1,\beta_1,m_1,p_1,\lambda,1,1)\\ &= R(\varepsilon,\delta+a\varepsilon,l_3-ym_1/3,t_3+yp_1/3,\alpha_1,\beta_1,m_1,p_1,\lambda,1,1)\\ &= R1(y,\varepsilon,\delta+a\varepsilon,l_3,t_3,\alpha_1,\beta_1,m_1,p_1),\end{aligned}$$

which is (4.23). □

Theorem 4.4. *Let α, β, m, p, and λ be as before with $\alpha m^2+\beta=p\lambda$, and let ε, δ, l, t be integers with $\varepsilon(p+t)+\delta(l+m)\equiv 1 \pmod 2$. Assume that $y=\pm 1$ with $y\equiv m \pmod 3$. Assume further that $3\mid\beta$ and $\gcd(3,m\lambda)=1$. Recall that $R1$ and $R2$ are defined by (4.17) and (4.2). Let α_1, β_1, m_1, and p_1 be another set parameters as in Corollary 3.3 and set $a:=(\alpha m-\alpha_1 m_1)/\lambda$. Then,*

$$R1(z,\varepsilon,\delta,l,t,\alpha,\beta,m,p)=R1(y,\delta,\varepsilon,l_1,t_11,\alpha\beta,\alpha m,\lambda), \tag{4.25}$$

where $l_1=t+(zp+\alpha my)/3$, $t_1=l-(zm+y\lambda)/3$, $z=\pm 1$ with $z\equiv -\lambda \pmod 3$. Moreover, if $3\mid\beta_1$ and $\gcd(3,\alpha_1 m_1)=1$, then

$$R1(y,\varepsilon,\delta,l,t,\alpha,\beta,m,p)=R1(y_1,\varepsilon,\delta+a\varepsilon,l_2,t_2,\alpha_1,\beta_1,m_1,p_1), \tag{4.26}$$

where $l_2=l-(ym-y_1m_1)/3$, $t_2=t+al+(yp-y_1p_1-aym)/3$, and $y_1=\pm 1$ with $y_1\equiv m_1 \pmod 3$. If $3\mid\alpha_1 m_1$, then

$$R1(y,\varepsilon,\delta,l,t,\alpha,\beta,m,p)=R2(\varepsilon,\delta+a\varepsilon,l_3,t_3,\alpha_1,\beta_1,m_1,p_1), \tag{4.27}$$

where $l_3=l+(1-ym)/3$, $t_3=t+al+y(p-am)/3$.

The proof of Theorem 4.4 is very similar to that of Theorem 4.3 and so we omit the proof.

5. Proofs of Entries 1.1–1.3

First, we record several pairs of identities which can easily be verified by (2.8):

$$\frac{f(-q^2,-q^8)}{f(-q,-q^9)}=\frac{f(-q^2,-q^3)}{\chi(-q)f(-q^{10})},\qquad \frac{f(-q^4,-q^6)}{f(-q^3,-q^7)}=\frac{f(-q,-q^4)}{\chi(-q)f(-q^{10})}, \tag{5.1}$$

$$\frac{f(-q^2,-q^3)}{f(q,q^4)}=\frac{\chi(-q)f(-q^4,-q^6)}{f(-q^5)},\qquad \frac{f(-q,-q^4)}{f(q^2,q^3)}=\frac{\chi(-q)f(-q^2,-q^8)}{f(-q^5)}, \tag{5.2}$$

$$\frac{f(-q^2,-q^6)}{f(q,q^7)}=\frac{\chi(-q)}{f(-q^8)}f\big(q^3,q^5\big),\qquad \frac{f(-q^2,-q^6)}{f(q^3,q^5)}=\frac{\chi(-q)}{f(-q^8)}f\big(q,q^7\big). \tag{5.3}$$

Recall that $g(q)$ and $h(q)$ are defined by (3.21). First, we prove Entry 1.1. By (4.23) with the set of parameters $\varepsilon=1$, $\delta=0$, $l=0$, $t=1$, $\alpha=6$, $\beta=76$, $m=3$, $p=5$ $(\lambda=26)$ and $\alpha_1=4$, $\beta_1=114$, $m_1=-2$, $p_1=5$ (with corresponding values of $z=1$, $\varepsilon=\delta=1$, $l=t=-1$), we have

$$R2(1,0,0,1,6,76,3,5)=R1(1,1,1,-1,-1,4,114,-2,5). \tag{5.4}$$

By Lemma 4.1, (5.1) with q replaced by q^{38} and by (2.13), we find that

$$\begin{aligned}
&R1(1,1,1,-1,-1,4,114,-2,5)\\
&\quad=q^{13/3}f\big(-q^{380}\big)\left(\frac{f(-q^{76},-q^{304})f(-q^8,-q^{32})}{f(-q^{38},-q^{342})}-q^6\frac{f(-q^{152},-q^{228})f(-q^{16},-q^{24})}{f(-q^{114},-q^{266})}\right)\\
&\quad=q^{13/3}\frac{1}{\chi(-q^{38})}\big(g\big(q^{38}\big)h\big(q^8\big)-q^6h\big(q^{38}\big)g\big(q^8\big)\big)\\
&\quad=q^{13/3}f\big(-q^8\big)f\big(-q^{76}\big)\big(G\big(q^{38}\big)H\big(q^8\big)-q^6H\big(q^{38}\big)G\big(q^8\big)\big).
\end{aligned} \tag{5.5}$$

By Lemma 4.2, (5.1) with q replaced by $-q^2$ and by (2.13), we similarly find that

$$\begin{aligned}
&R2(1,0,0,1,6,76,3,5)\\
&\quad=q^{13/3}f\big(-q^{20}\big)\left(\frac{f(-q^8,-q^{12})f(-q^{304},-q^{456})}{f(q^6,q^{14})}+q^{30}\frac{f(-q^4,-q^{16})f(-q^{152},-q^{608})}{f(q^2,q^{18})}\right)\\
&\quad=\frac{1}{\chi(q^2)}\big(h\big(-q^2\big)g\big(q^{152}\big)+q^{30}g\big(-q^2\big)h\big(q^{152}\big)\big)\\
&\quad=f\big(-q^4\big)f\big(-q^{152}\big)\big(H\big(-q^2\big)G\big(q^{152}\big)+q^{30}G\big(-q^2\big)H\big(q^{152}\big)\big).
\end{aligned} \tag{5.6}$$

By (5.4)–(5.6), we have

$$\frac{G(q^{38})H(q^8)-q^6H(q^{38})G(q^8)}{H(-q^2)G(q^{152})+q^{30}G(-q^2)H(q^{152})}=\frac{f(-q^4)f(-q^{152})}{f(-q^8)f(-q^{76})}=\frac{\chi(-q^4)}{\chi(-q^{76})}, \tag{5.7}$$

which is (1.4) with q replaced by q^2.

Next, we simultaneously prove Entries 1.2 and 1.3. By (4.25) with the set of parameters $z=1$, $\varepsilon=1$, $\delta=0$, $l=t=1$, $\alpha=11$, $\beta=9$, $m=1$ and $p=4$ $(\lambda=5)$, we find that

$$R1(1,1,0,1,1,11,9,1,4)=R1(1,0,1,6,-1,1,99,11,5). \tag{5.8}$$

By Lemma 4.1, and by (5.1) with q replaced by q^{33}, we find that

$$\begin{aligned}
&R1(1,0,1,6,-1,1,99,11,5)\\
&\quad=q^{9/4}f\big(-q^{330}\big)\left(\frac{f(-q^{66},-q^{264})f(-q^4,-q^6)}{f(-q^{33},-q^{297})}+q^7\frac{f(-q^{132},-q^{198})f(-q^2,-q^8)}{f(-q^{99},-q^{231})}\right)\\
&\quad=q^{9/4}\frac{1}{\chi(-q^{33})}\big(g\big(q^2\big)g\big(q^{33}\big)+q^7h\big(q^2\big)h\big(q^{33}\big)\big).
\end{aligned} \tag{5.9}$$

By Lemma 4.1, and by (5.3) with q replaced by q^3, we similarly find that

$$\begin{aligned} &R1(1,1,0,1,1,11,9,1,4) \qquad (5.10)\\ &= q^{9/4} f\left(-q^{24},-q^{48}\right)\left(\frac{f(-q^6,-q^{18})f(q^{33},q^{55})}{f(q^9,q^{15})} - q^4\frac{f(-q^6,-q^{18})f(q^{11},q^{77})}{f(q^3,q^{21})}\right)\\ &= q^{9/4}\chi\left(-q^3\right)\left\{f\left(q^3,q^{21}\right)f\left(q^{33},q^{55}\right) - q^4 f\left(q^9,q^{15}\right)f\left(q^{11},q^{77}\right)\right\}. \qquad (5.11) \end{aligned}$$

Using (2.17), we easily find that

$$\psi\left(q^3\right)\psi\left(-q^{11}\right) - \psi\left(-q^3\right)\psi\left(q^{11}\right) = 2q^3\left\{f\left(q^6,q^{42}\right)f\left(q^{66},q^{110}\right) - q^8 f\left(q^{18},q^{30}\right)f\left(q^{22},q^{154}\right)\right\}. \qquad (5.12)$$

Now in (5.9) and (5.11), we replace q by q^2 and use (5.8) and (5.12) to conclude that

$$2q^3\left(g\left(q^4\right)g\left(q^{66}\right) + q^{14}h\left(q^4\right)h\left(q^{66}\right)\right) = \chi\left(-q^6\right)\chi\left(-q^{66}\right)\left\{\psi\left(q^3\right)\psi\left(-q^{11}\right) - \psi\left(-q^3\right)\psi\left(q^{11}\right)\right\}. \qquad (5.13)$$

In what follows, $J(q)$ will denote an arbitrary power series, usually not the same with each appearance. By (2.15) with $n=3$ in each instance,

$$\begin{aligned} g(q) &= f\left(-q^2,-q^3\right) = f\left(-q^{21},-q^{24}\right) - q^2 f\left(-q^9,-q^{36}\right) - q^3 f\left(-q^6,-q^{39}\right)\\ &= J\left(q^3\right) - q^2 h\left(q^9\right), \qquad (5.14)\\ h(q) &= f\left(-q,-q^4\right) = f\left(-q^{18},-q^{27}\right) - qf\left(-q^{12},-q^{33}\right) - q^4 f\left(-q^3,-q^{42}\right)\\ &= g\left(q^9\right) - qJ\left(q^3\right), \qquad (5.15)\\ \psi(q) &= f\left(q^3,q^6\right) + q\psi\left(q^9\right) = J\left(q^3\right) + q\psi\left(q^9\right), \qquad (5.16) \end{aligned}$$

where, in the last application of (2.15), we set $a=1$ and $b=q$ and used (2.4) and (2.10). By (5.16),

$$\begin{aligned} &\psi\left(q^3\right)\psi\left(-q^{11}\right) - \psi\left(-q^3\right)\psi\left(q^{11}\right) \qquad (5.17)\\ &= \left\{J\left(q^3\right) - q^{11}\psi\left(-q^{99}\right)\right\}\psi\left(q^3\right) - \left\{J\left(q^3\right) + q^{11}\psi\left(q^{99}\right)\right\}\psi\left(-q^3\right)\\ &= J\left(q^3\right) - q^{11}\left\{\psi\left(q^3\right)\psi\left(-q^{99}\right) + \psi\left(-q^3\right)\psi\left(q^{99}\right)\right\}. \qquad (5.18) \end{aligned}$$

Similarly, by (5.14) and (5.15) with q replaced by q^2, we find that

$$g\left(q^4\right)g\left(q^{66}\right) + q^{14}h\left(q^4\right)h\left(q^{66}\right) = J\left(q^3\right) - q^8\left\{h\left(q^{36}\right)g\left(q^{66}\right) - q^6 g\left(q^{36}\right)h\left(q^{66}\right)\right\}. \qquad (5.19)$$

From these last two equalities and (5.13), we conclude that

$$2\left\{h\left(q^{12}\right)g\left(q^{22}\right) - q^2 g\left(q^{12}\right)h\left(q^{22}\right)\right\} = \chi\left(-q^2\right)\chi\left(-q^{22}\right)\left(\psi(q)\psi\left(-q^{33}\right) + \psi(-q)\psi\left(q^{33}\right)\right). \qquad (5.20)$$

Next, by (4.25) with the set of parameters $z=-1$, $\varepsilon=0$, $\delta=1$, $l=0$, $t=1$, $\alpha=11$, $\beta=9$, $m=1$ and $p=5$ ($\lambda=4$), we find that

$$R1(-1,0,1,0,1,11,9,1,5) = R1(1,1,0,3,-1,1,99,11,4). \qquad (5.21)$$

By Lemma 4.2, and by (5.3) with q replaced by q^{33}, we find that

$$
\begin{aligned}
&R1(1,1,0,3,-1,1,99,11,4)\\
&\quad = q^{3/4} f\left(-q^{264}\right)\left(\frac{f(-q^{66},-q^{198}) f(q^3,q^5)}{f(q^{33},q^{231})} - q^{17}\frac{f(-q^{66},-q^{198}) f(q,q^7)}{f(q^{99},q^{165})}\right)\\
&\quad = q^{3/4}\chi\left(-q^{33}\right)\left\{f\left(q^3,q^5\right) f\left(q^{99},q^{165}\right) - q^{17} f\left(q,q^7\right) f\left(q^{33},q^{231}\right)\right\}.
\end{aligned}
\tag{5.22}
$$

By Lemma (4.1), and by (5.1) with q replaced by q^3, we similarly find that

$$
\begin{aligned}
&R1(-1,0,1,0,1,11,9,1,5)\\
&\quad = q^{3/4} f\left(-q^{30}\right)\left(\frac{f(-q^6,-q^{24}) f(-q^{44},-q^{66})}{f(-q^3,-q^{27})} + q^5\frac{f(-q^{12},-q^{18}) f(-q^{22},-q^{88})}{f(-q^9,-q^{21})}\right)\\
&\quad = q^{3/4}\frac{1}{\chi(-q^3)}\left\{g\left(q^3\right) g\left(q^{22}\right) + q^5 h\left(q^3\right) h\left(q^{22}\right)\right\}.
\end{aligned}
\tag{5.23}
$$

Arguing as before from (5.21)–(5.23) and (2.17), we conclude that

$$
2\left\{g\left(q^6\right) g\left(q^{44}\right) + q^{10} h\left(q^6\right) h\left(q^{44}\right)\right\} = \chi\left(-q^6\right)\chi\left(-q^{66}\right)\left(\psi(q)\psi\left(-q^{33}\right) + \psi(-q)\psi\left(q^{33}\right)\right). \tag{5.24}
$$

Considering the 3-dissection of both sides of (5.24), one similarly obtains

$$
2q^3\left(h\left(q^2\right) g\left(q^{132}\right) - q^{26} g\left(q^2\right) h\left(q^{132}\right)\right) = \chi\left(-q^2\right)\chi\left(-q^{22}\right)\left(\psi\left(q^3\right)\psi\left(-q^{11}\right) - \psi\left(-q^3\right)\psi\left(q^{11}\right)\right). \tag{5.25}
$$

Comparing (5.12) and (5.25), and using (2.13), we see that Entry 1.2 is proved. Similarly, the identities (5.24) and (5.20) imply Entry 1.3. In [2], we showed that Entries 1.2 and 1.3 are equivalent to each other.

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